PHILOSOPHICAL TRANSACTIONS.

XVI. *Of the Rotatory Motion of a Body of any Form whatever, revolving, without Reſtraint, about any Axis paſſing through its center of Gravity. By Mr.* John Landen, *F.R.S.*

Read March 17, 1785.

A SPHERICAL body, uniformly denſe, it is obvious, will, if made to revolve freely about any axis paſſing through its center, continue to revolve about the ſame axis; and, by what I have ſhewn in the *Philoſophical Tranſactions* for the year 1777, it appears, that a *cylinder* of uniform denſity, whoſe length is to its radius as $\sqrt{3}$ to 1, will do the ſame. It likewiſe appears, by my *Mathematical Memoirs*, that

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that a *cone*, a *conoid*, a *priſm*, or a *pyramid*, &c. of certain dimenſions, will have the like property of continuing, without any reſtraint, to revolve about any axis paſſing through its center of gravity.

When the axis, about which a body may be made to revolve, is not a permanent one, the centrifugal force of its particles will diſturb its rotatory motion, ſo as to cauſe it to change its axis of rotation (and conſequently its poles) every inſtant, and endeavour to revolve about a new one: and I cannot think it will be deemed an unintereſting propoſition to determine in what track, and at what rate, the poles of ſuch momentary axis will be varied in any body whatever; as, without the knowledge to be obtained from the ſolution of ſuch problem, we cannot be certain whether the earth, or any other planet, may not, from the *inertia* of its own particles, ſo change its momentary axis, that the poles thereof ſhall approach nearer and nearer to the preſent equator, or whether the evagation of the momentary poles, ariſing from that cauſe, will not be limited by ſome known leſſer circle. Which certainly is an important conſideration in aſtronomy; eſpecially now that branch of ſcience is carried to great perfection, and the acute aſtronomer endeavours to determine the motions of the heavenly bodies with the greateſt exactneſs poſſible.

I do not know that the problem has before been ſolved by any mathematician in theſe kingdoms; but I am aware that it has been conſidered by ſome gentlemen, very eminent for their mathematical knowledge, in other nations. The ſolutions of it, given by the celebrated M. LEONHARD EULER and M. D'ALEMBERT, I have ſeen: and we learn from what the laſt mentioned gentleman has ſaid, in his *Opuſcules Mathematiques*, that a ſolution of it, inveſtigated by M. JOHN ALBERT EULER (after a method ſimilar to his father's) obtained the prize given by the Academy of Sciences in the year 1761. The concluſions deduced by thoſe very learned gentlemen differing greatly from mine made me ſuſpect, for ſome time, that I had ſomewhere erred in my inveſtigation, and induced me to reviſe my proceſs again and again with the greateſt circumſpection. At length my ſcrutiny has ſo removed my doubts, that, being well aſſured of the truth of my theory, I now beg leave to preſent it to the Royal Society; preſuming that it will be found not unworthy of the notice of ſuch readers, as are curious in contemplating the various motions which bodies may naturally have, in conſequence of inſtantaneous or continued impulſe.

In the *Philoſophical Tranſactions* referred to above, I gave a ſpecimen of this theory, as far as it relates to the motion of *a ſpheroid* and *a cylinder*. The improvements I have ſince made in it, enable me now to extend it to the motion of *any body whatever*, how irregular ſoever its form may be.

What I here infer therefrom will be found to differ very materially from the deductions in the ſolutions given by the gentlemen above-mentioned. They repreſent the angular velocity, and the momentum of rotation of the revolving body, as always *variable*, when the axis about which it has a tendency to revolve is a momentary one, except in a particular caſe. By my inveſtigation it appears, that the angular velocity and the momentum of rotation will always be *invariable* in any revolving body, though the axis about which it endeavours to revolve be continually varied; and the tracks of the varying poles upon the ſurface of the body are thereby determined with great facility.

It is not only obſervable, that the tracks which the varying poles take, in the ſurface of any revolving body, are ſuch that its momentum of rotation may continue the ſame whilſt its angular velocity continues the ſame; but it may be obſerved, that, in any given body, there is only one ſuch track which a momentary pole can purſue from any given point.

If the angular velocity and the momentum of rotation of a revolving body were to vary according to the computations adverted to above, it would follow, that a body might acquire an increaſe of force from its own motion, without being any way affected by any other body whatever, as the ſame percuſſive force, applied at the ſame diſtance from the momentary axis, would not always deſtroy the rotatory motion of the body, which ſurely cannot poſſibly be true. From the principles or laws of motion, which I conſider as undoubtedly true (and which indeed are no other than the common principles of mechanics), I conclude that a revolving body, not affected by any external impulſe, can no more acquire an increaſe in its momentum of rotation, than any other body, moving freely, can acquire an increaſe in its momentum, or quantity of motion, in any given direction, without being impelled by gravity or ſome other force. And the truth of this concluſion (which is hereinafter proved by other reaſoning) may be eaſily inferred from the property of the lever; ſeeing that the joint centrifugal force of the particles of the revolving body (which is the *only* diſturbing force) has no tendency to accelerate or retard their motion about the momentary axis, but only to alter the poſition of ſuch axis, the direction in which that force acts being always in a plane wherein that axis will be found.

By the theory explained in this paper, it appears that a *parallelopipedon* may always be conceived of ſuch dimenſions, that being, by ſome force or forces, made to revolve about an axis, paſſing through its center of gravity, with a certain angular velocity, it ſhall move exactly in the ſame manner as any other given body will move, if made to revolve, by the ſame force or forces, about an axis paſſing through its center of gravity; the quantity of matter (as well as the initial angular velocity) being ſuppoſed the ſame in both bodies; and due regard being had, in the application of the moving force or forces, to the correſponding planes in the bodies. Therefore, as we may from thence always aſſign the dimenſions of a *parallelopipedon* that ſhall be affected exactly in the ſame manner as any other given body will be affected, as well with regard to the centrifugal force of the reſpective particles of the bodies, as to the action of equal percuſſive forces, or oſcillation; it will, after ſhewing how the dimenſions of ſuch *parallelopipedon* may be computed, be only neceſſary, in inveſtigating the propoſition under conſideration, to determine the tracks and velocities of the poles of the momentary axis, about which any *parallelopipedon* may be made to revolve.

Firſt then to find ſuch parallelopipedon (P), that, with reſpect to the action of ſuch forces as are mentioned above, it may be affected exactly in the ſame manner as any other given body (Q). Let it be conſidered that G (tab. X. fig. 1.) being the center of gravity, N a point of ſuſpenſion, and O the correſponding center of oſcillation or percuſſion, the rectangle $GN \times GO$ will be an invariable quantity, the direction NGO continuing the ſame; and that a cylindric ſurface being deſcribed, ſuch that the center of the middle circular ſection thereof ſhall be G, and radius $= \sqrt{GN \times GO}$, and whoſe axis ſhall be perpendicular to the plane wherein the line NGO is ſuppoſed to be impelled to move; if all the matter in the body were placed any where in that ſurface, ſo that G ſhould be the center of gravity of the matter ſo placed, any given force or forces, acting on the body in the plane juſt now mentioned, would cauſe the line NGO in the body to move exactly in the ſame manner as it would move, if it were carried with the matter placed in the ſaid ſurface (as before-mentioned) after having been put in motion by the action of the ſame force or forces. Moreover, let it be conſidered, that there will at leaſt be three permanent axes of rotation in the body Q, at right angles to each other (as I have proved in my *Mathematical Memoirs*); and that, ſuppoſing NGO to coincide with thoſe three axes in three ſucceſſive caſes wherein the matter in Q ſhall, in each caſe, be conceived to be placed in a cylindric ſurface as deſcribed above, we may conceive it poſſible ſo to place the matter of the body, that all of it ſhall be in each of thoſe three ſurfaces, and G ſtill continue its center of gravity. And, a computation being made accordingly, it appears, that the matter of the body Q muſt be placed, in equal quantities, at each of the eight angular points of a *parallelopipedon* (R) whoſe dimenſions (length, breadth, and thickneſs) ſhall be $\sqrt{2d^2+2f^2-2e^2}$, $\sqrt{2e^2+2f^2-2d^2}$, and $\sqrt{2d^2+2e^2-2f^2}$; d, e, and f, being the three values of $\sqrt{GN \times GO}$, when NGO is ſucceſſively a permanent axis of rotation, with reſpect to the body Q, in three directions at right angles to each other.

If Q were a parallelopipedon, it may be eaſily proved, that its dimenſions muſt be $\sqrt{6d^2+6f^2-6e^2}$, $\sqrt{6e^2+6f^2-6a^2}$, and $\sqrt{6d^2+6e^2-6f^2}$, that the correſponding parallelopipedon, at the angular points whereof the matter of Q is conceived to be placed as above, may have the ſame dimenſions as thoſe which we have found our parallelopipedon R muſt have.

Whence we may infer, that the parallelopipedon (P), which we propoſed to find, muſt have the dimenſions laſt written; namely, length, breadth, and thickneſs, reſpectively equal to $\sqrt{6d^2+6f^2-6e^2}$, $\sqrt{6e^2+6f^2-6d^2}$, and $\sqrt{6d^2+6e^2-6f^2}$; which may be confirmed by a more ſtrict demonſtration founded on the principles made uſe of in my *fourth Memoir.* For it appears by what is there proved, that the centrifugal forces of the particles of any revolving body, in two directions at right angles to each other, may be expreſſed in terms of A, B, K, and variable quantities ſhewing the poſition of the momentary axis; and that, in a parallelopipedon whoſe dimenſions (length, breadth, and thickneſs) are a, b, k; and whoſe maſs, or content, is $=$ M; A will be $=\frac{Ma^2}{12}$, $B=\frac{Mb^2}{12}$, and $K=\frac{Mk^2}{12}$. If therefore a be $=\sqrt{6d^2+6f^2-6e^2}$, $b=\sqrt{6e^2+6f^2-6d^2}$, and $k=\sqrt{6d^2+6e^2-6f^2}$; in ſuch body,

$$\text{A will be}=\frac{M}{2}\times\overline{d^2+f^2-e^2},$$

$$B \quad =\frac{M}{2}\times\overline{e^2+f^2-d^2},$$

$$K \quad =\frac{M}{2}\times\overline{d^2+e^2-f^2}.$$

But, in any body whatever,

$$M\times d^2 \text{ is } = \text{the ſum of all the } \overline{x^2+z^2}\times p,$$

$$M\times e^2 \quad = \text{the ſum of all the } \overline{y^2+z^2}\times p,$$

$$M\times f^2 \quad = \text{the ſum of all the } \overline{x^2+y^2}\times p,$$

and $\frac{M}{2}\times\overline{d^2+e^2+f^2}=$ the ſum of all the $\overline{x^2+y^2+z^2}\times p$: x, y, and z correſponding to the place of the particle p in the body; x being meaſured from the center of gravity upon a permanent axis of rotation, y at right angles to x, and z at right angles to y in a plane to which the ſaid axis is perpendicular. Therefore,

A, which is = the ſum of all the $x^2 \times p$, will be $= \frac{M}{2} \times \overline{d^2+f^2-e^2}$,

B, = the ſum of all the $y^2 \times p$, $= \frac{M}{2} \times \overline{e^2+f^2-d^2}$,

K, = the ſum of all the $z^2 \times p$, $= \frac{M}{2} \times \overline{d^2+e^2-f^2}$,

Hence it is evident, that d, e, and f being determined from any body whatever, the values of A, B, and K will be the ſame in that body as in our parallelopipedon P; and that the centrifugal forces of the particles will be the ſame in both bodies. Conſequently, their motions about ſucceſſive momentary axes (whoſe poles are varied by the perturbation ariſing from thoſe forces), will be the ſame in both bodies; their initial angular velocities being the ſame; as well as the poſition of their initial momentary axes, with reſpect to the correſpondent permanent axes of rotation in each body.

Let us now proceed to find how any *parallelopipedon* will revolve about ſucceſſive momentary axes paſſing through its center of gravity: by which means, with the help of the theorem juſt now inveſtigated, we ſhall be enabled to define how any body whatever will revolve about ſuch axes; which is the chief purpoſe of this diſquiſition.

Fig. 2. and 3. The length, breadth, and thickneſs of the revolving *parallelopipedon* (P) being $2d$, $2c$, and $2b$, conceive a ſpherical ſurface without matter, whoſe center is the center of gravity of the body P, to be carried about with that body during its motion; and let the ſaid ſurface be orthographically projected, ſo that the radius upon which b is meaſured may be repreſented by AB; the radius upon which d is meaſured may be repreſented by AD; and the radius AC, upon which c is meaſured, may be projected into the central point A. Let P be the momentary pole, and PQ the continuation of the great circle CP. Let a denote the radius AB ($=$ AD); g and γ the ſine and coſine of the arc CP; s and t the ſine and coſine of the arc BQ, to the ſame radius a; e the angular velocity of the body and ſpherical ſurface, meaſured at the diſtance a from the momentary axis; and M the maſs or content of the parallelopipedon ($=8bcd$).

Then the motive force E, urging the pole P towards Q, will (by what I have proved in my *Mathematical Memoirs*) be $=\frac{\mathrm{M}e^2g\gamma}{3a^7} \times \overline{\mathrm{D}s^2 - \mathrm{C}a^2}$; and the motive force $\overset{\shortparallel}{\mathrm{E}}$, urging the ſame pole in a direction Po, at right angles to that in which E acts, $=\frac{\mathrm{M}e^2g}{3a^6} \times \mathrm{D}st$; C and D being equal to $c^2 - b^2$ and $d^2 - b^2$ reſpectively. Let Pq be to Po as E to $\overset{\shortparallel}{\mathrm{E}}$; complete the parallelogram oPqr, and draw the diagonal Pr. This laſt mentioned line will (by what I have ſhewn in the *Philoſophical Tranſactions* for the year 1777) be perpendicular to the tangent to the polar track at P. Therefore $\mathrm{P}p\overset{\shortparallel}{p}\overset{\shortparallel\!\vert}{p}$ being the projection of that track, and Pp an indefinitely ſmall particle thereof; if pu be perpendicular to PuA, and the quantities $d^2 - c^2$, $c^2 - b^2$, be not negative; $\frac{\gamma}{a} \times \overline{\mathrm{D}s^2 - \mathrm{C}a^2}$ will be to Dst (as Pq to Po) as pu to Pu, the triangles Por and Pup being ſimilar, and $or =$ Pq. But with reſpect to our ſpherical ſurface, pu will be to Pu as $\frac{g\dot{s}}{t}$ to $-\frac{a\dot{g}}{\gamma}$; therefore, $\overline{\mathrm{C}a^2 - \mathrm{D}s^2} \times \dot{g}$ will be $=\mathrm{D}gs\dot{s}$, and $\frac{\dot{g}}{g} = \frac{\mathrm{D}s\dot{s}}{\mathrm{C}a^2 - \mathrm{D}s^2}$. Whence, by taking the fluents, we have $s^2 = a^2 \times \frac{Dm^2 - C\gamma^2}{Dg^2}$, and $t^2 = a^2 \times \frac{Dn^2 - B\gamma^2}{Dg^2}$; *m* and *n* denoting the values of *s* and *t*, when *g* is $= a$ and $\gamma = 0$; and B being put to denote the difference $D - C = d^2 - c^2$.

If now β and δ be put to denote the coſines of BP and DP to the radius *a*, we ſhall, from what is done above, have

$$\beta = \frac{gt}{a} = \frac{\sqrt{Dn^2 - B\gamma^2}}{D^{\frac{1}{2}}},\ \delta = \frac{gs}{a} = \frac{\sqrt{Dm^2 - C\gamma^2}}{D^{\frac{1}{2}}};$$

$$\beta^2 + \gamma^2 + \delta^2 = a^2,\ \beta\dot{\beta} + \gamma\dot{\gamma} + \delta\dot{\delta} = 0;$$

$$b^2\beta^2 + c^2\gamma^2 + d^2\delta^2 = b^2n^2 + d^2m^2, \text{ and } b^2\beta\dot{\beta} + c^2\gamma\dot{\gamma} + d^2\delta\dot{\delta} = 0.$$

Drawing AR ſo that $D^{\frac{1}{2}} \times$ ſine of BR ſhall be $= C^{\frac{1}{2}}a$, it is *very remarkable*, that the momentary pole (P) will run round about the point B, or about the point D, in the ſpherical ſurface, according as the initial pole ſhall be in the part BCR or DCR of the ſaid ſurface; that is, according as Dm^2 is leſs or greater than Ca^2: and that, if the initial pole (P) be any where in the great circle CR, the momentary pole, keeping in the arc of that circle, will continually approach nearer and nearer to the point C in the ſurface of the ſphere; but, by what follows, we ſhall find that it never can arrive at that point in any finite time!

The equation of the track of the pole in the projection to which we have hitherto referred will, it is now obvious, be $y^2 = \frac{B}{C} \times \overline{x^2 + \frac{Ca^2 - Dm^2}{B}}$; *x*, meaſured from the center A upon AD, being $= \delta$; and *y*, at right angles thereto, $= \beta$.

If C be $= 0$ (that is, if *c* be $= b$), *x* will be equal to the invariable quantity *m*; the projected track, a *right line* parallel to AB; and the track upon the ſurface of the ſphere, a *leſſer circle* in a plane parallel to the plane of the great circle BC.

If C be$=$D, y will be equal to the invariable quantity n; the projected track, a *right line* parallel to AD; and the track on the ſurface of the ſphere, a *leſſer circle* in a plane parallel to the plane of the great circle CD.

If Dm^2 be$=Ca^2$ the projected track will be the *right line* AR, and $y=\overline{\frac{B}{C}}\Big|^{\frac{1}{2}}\times x$; the track upon the ſurface of the ſphere being the *great circle* CR.

In all other caſes in this projection, the track will be an *hyperbola* whoſe center is A, ſemi axis $Aa=\overline{\frac{Ca^2 \backsim Dm^2}{B}}\Big|^{\frac{1}{2}}$, and the other ſemi-axis $=\overline{\frac{Ca^2 \backsim Dm^2}{C}}\Big|^{\frac{1}{2}}$; the right line AR being always an *aſypmtote.*

Fig. 4. When the track is projected on a plane ACD, to which the radius AB is perpendicular (the point D being the vertex as before) the equation thereof will be $y^2=\frac{D}{C}\times\overline{m^2-x^2}$; x, meaſured from the center A upon AD, being$=\delta$ (as before); and y, at right angles thereto$=\gamma$. This projection of the track of the pole will therefore always be an *ellipſis* a b (or a *circle*) whoſe center is A; ſemi-axis A a$=m$; and the other ſemi-axis $=\overline{\frac{D}{C}}\Big|^{\frac{1}{2}}\times m$: except c be$=b$; in which caſe the projected track will be a *right line* a b parallel to AC.

Fig. 5. Moreover, the equation of the track projected on the plane ABC, to which the radius AD is perpendicular, will be $y^2=\frac{D}{B}\times\overline{n^2-x^2}$; x, meaſured from the center A upon AB, being$=\beta$; and y, at right angles thereto, $=\gamma$. The track of the pole in this projection will therefore always be an *ellipſis*

a b (or a *circle*) whoſe center is A; ſemi-axis A a$=n$; and the other ſemi-axis $=\overline{\frac{D}{B}}\Big|^{\frac{1}{2}} \times n$; except c be$=d$; in which caſe the projected track will be a *right line* a b parallel to AC.

With regard to the permanent axes of rotation of our parallelopipedon, it appears, by my *Mathematical Memoirs*, that if two of its dimenſions be equal (that is, when the body is a *ſquare priſm*), any line paſſing through the center of gravity of the body, in a plane to which the other dimenſion is perpendicular, will be a permanent axis of rotation; as will the line paſſing through that center, at right angles to that plane. If all the three dimenſions be equal (that is, when the body is a *cube*), any line whatever paſſing through the center of gravity of the body will be a permanent axis of rotation.

It is obſervable, that the momentum of rotation of the body, about the momentary axis, is found by computation always $= \frac{e}{a^2} \times \overline{b^2m^2 + c^2a^2 + d^2n^2}$, e denoting the angular velocity. But $\frac{f}{a^2} \times \overline{b^2m^2 + c^2a^2 + d^2n^2}$ is the initial momentum of rotation. Therefore, conſidering the momentum of rotation as invariable, the angular velocity will be invariable, e being always$=f$, which here denotes the initial angular velocity.

Our next buſineſs is to find the length of the track deſcribed by the momentary pole (P), upon the ſpherical ſurface; and the velocity of the pole in that track.

Fig. 2, 3. It appearing, that the motive force E is$=\frac{Me^2}{3a^5} \times \overline{Dm^2 - Ca^2} \times \frac{\gamma}{g}$, and the motive force $\overset{||}{E} = \frac{Me^2}{3a^4g} \times \sqrt{Dm^2 - C\gamma^2} \times \sqrt{Dn^2 - B\gamma^2}$; we find $F = \sqrt{E^2 + \overset{||}{E}{}^2}$ (the force compounded of thoſe two forces) $= \frac{Me^2}{3a^5} \times \sqrt{D^2m^2n^2 - BCa^2\gamma^2}$; and, F being to E as *a* to the ſine of the angle *p*P*u*, it follows, that the ſine of *p*P*u* will be $= \frac{\overline{\mathrm{D}m^2 - \mathrm{C}a^2} \times a\gamma}{g\sqrt{\mathrm{D}^2m^2n^2 - \mathrm{BC}a^2\gamma^2}}$, and its coſine $= \frac{a^2\sqrt{\mathrm{D}m^2 - \mathrm{C}\gamma^2} \times \sqrt{\mathrm{D}n^2 - \mathrm{B}\gamma^2}}{g\sqrt{\mathrm{D}^2m^2n^2 - \mathrm{BC}a^2\gamma^2}}$. Therefore, that coſine being to radius as $\left(\frac{a\dot{\gamma}}{g}\right)$ the fluxion of the arc PQ to ($\dot{z}$) the fluxion of the polar track on the ſpherical ſurface, $\dot{z}$ will be $= \frac{\dot{\gamma}\sqrt{\mathrm{D}^2m^2n^2 - \mathrm{BC}a^2\gamma^2}}{\sqrt{\mathrm{D}m^2 - \mathrm{C}\gamma^2} \times \sqrt{\mathrm{D}n^2 - \mathrm{B}\gamma^2}}$. Now, P*p*LN being a quadrant of a great circle (touching the ſaid polar track at P), and NAN a diameter of that circle; if we put *w* to denote the diſtance of any particle *(p)* of the parallelopipedon from that diameter, and G to denote the accelerative force of any ſuch particle when *w* is $= a$; the motive force F $\left(= \frac{\mathrm{M}e^2}{3a^5} \times \sqrt{\mathrm{D}^2m^2n^2 - \mathrm{BC}a^2\gamma^2}\right)$, computed above, will be $= \frac{\mathrm{G}}{a^2} \times$ *the ſum of all the* $w^2 \times p$; which ſum, by computation, is found $= \frac{\mathrm{M}}{3} \times \frac{\overline{d^2 + b^2} \cdot \mathrm{D}^2m^2n^2 - \overline{b^2m^2 + c^2a^2 + d^2n^2} \cdot \mathrm{BC}\gamma^2}{\mathrm{D}^2m^2n^2 - \mathrm{BC}a^2\gamma^2}$. Conſequently, G will be $= \frac{e^2}{a^3} \times \frac{\overline{\mathrm{D}^2m^2n^2 - \mathrm{BC}a^2\gamma^2}|^{\frac{3}{2}}}{\overline{d^2 + b^2} \cdot \mathrm{D}^2m^2n^2 - \overline{b^2m^2 + c^2a^2 + d^2n^2} \cdot \mathrm{BC}\gamma^2}$. But, by what I have done in the *Philoſophical Tranſactions* for the year 1777, $\frac{a\mathrm{G}}{e}$ will be $= v =$ the velocity wherewith the momentary pole changes its place in the ſpherical ſurface to which it is referred. Therefore,

v will be $= \frac{e}{a^2} \times \frac{\overline{\mathrm{D}^2m^2n^2 - \mathrm{BC}a^2\gamma^2}|^{\frac{3}{2}}}{\overline{d^2 + b^2} \cdot \mathrm{D}^2m^2n^2 - \overline{b^2m^2 + c^2a^2 + d^2n^2} \cdot \mathrm{BC}\gamma^2}$; and $\frac{\dot{z}}{v} = \dot{\mathrm{T}}$, the fluxion of the time $= \frac{a^2\dot{\gamma}}{e} \times \frac{\overline{d^2 + b^2} \cdot \mathrm{D}^2m^2n^2 - \overline{b^2m^2 + c^2a^2 + a^2n^2} \cdot \mathrm{BC}\gamma^2}{\sqrt{\mathrm{D}m^2 - \mathrm{C}\gamma^2} \times \sqrt{\mathrm{D}n^2 - \mathrm{B}\gamma^2} \times \mathrm{D}^2m^2n^2 - \mathrm{BC}a^2\gamma^2}$

$$= \frac{b^2m^2+c^2a^2+d^2n^2}{e} \times \left\{ \begin{array}{l} \dfrac{\dot{\gamma}}{\sqrt{Dm^2-C\gamma^2}\times\sqrt{Dn^2-B\gamma^2}} + \\ \dfrac{\dfrac{Dm^2-Ca^2}{b^2m^2+c^2a^2+d^2n^2}\times D^2m^2n^2\dot{\gamma}}{\sqrt{Dm^2-C\gamma^2}\times\sqrt{Dn^2-B\gamma^2}\times\overline{D^2m^2n^2-BCa^2\gamma^2}} \end{array}\right.$$: which,

when Dm^2 is $=Ca^2$, becomes $= \frac{\overline{d^2+b^2}\,.\,a^2\dot{\gamma}}{e\sqrt{BC}\,.\,\overline{a^2-\gamma^2}}$.

It is evident, that $\frac{d^2+b^2}{2e\sqrt{BC}}\times a\times$ hyp. log. of $\frac{a+\gamma}{a-\gamma}$, the value of T in that particular caſe, will be *infinite* when γ is $=a$; and this concluſion agrees with what is ſaid above reſpecting the motion of the momentary pole along the great circle CR (fig. 2. and 3.).

I have not found, that the value of T will, in general, be aſſigned by the *arcs of the conic ſections*; but my Tables * ſhew, that it will be ſo aſſigned when Dm^2 is $=Ba^2$, and in ſome other particular caſes.

We have ſtill to inveſtigate the track of the momentary pole in the *immoveable* concave ſpherical ſurface, which we muſt conceive to ſurround our *moveable* convex ſpherical ſurface, ſuppoſing the center of both thoſe ſurfaces to coincide with the centers of gravity of our parallelopipedon: which central point is always in this diſquiſition ſuppoſed at reſt.

Let AL be the projection of part of a great circle CL, at right angles to the great circle P*p*LN; then will the ſine of the arc CL be $= \frac{\overline{Dm^2-Ca^2}\times\gamma}{\sqrt{D^2m^2n^2-BCa^2\gamma^2}}$; its coſine $=$ $\frac{D^{\frac{1}{2}}\sqrt{Da^2m^2n^2-\overline{Dm^4-Ca^2m^2+Ca^2n^2}\,.\,\gamma^2}}{\sqrt{D^2m^2n^2-BCa^2\gamma^2}}$; and, the fluxion of that ſine being $= \frac{\overline{Dm^2-Ca^2}\times D^2m^2n^2\dot{\gamma}}{\overline{D^2m^2n^2-BCa^2\gamma^2}|^{\frac{3}{2}}}$, the fluxion of that arc (CL) will be

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* *Mathematical Memoirs*, publiſhed in 1780.

$= \frac{\overline{Dm^2 - Ca^2} \times D^{\frac{3}{2}} a m^2 n^2 \dot{\gamma}}{\overline{D^2m^2n^2 - BCa^2\gamma^2} \times \sqrt{Da^2m^2n^2 - \overline{Dm^4 - Ca^2m^2 + Ca^2n^2} \,.\, \gamma^2}}$. Conſequently, the ſine of the arc PL being $= \frac{a^2\sqrt{Dm^2 - C\gamma^2} \times \sqrt{Dn^2 - B\gamma^2}}{D^{\frac{1}{2}}\sqrt{Da^2m^2n^2 - \overline{Dm^4 - Ca^2m^2 + Ca^2n^2} \,.\, \gamma^2}}$; and this ſine being to radius as the fluxion of the arc CL to the meaſure of the angle of contact of the polar track on the *moveable* ſpherical ſurface with a great circle, we find that meaſure

$$= \frac{\overline{Dm^2 - Ca^2} \times D^2m^2n^2\dot{\gamma}}{\sqrt{Dm^2 - C\gamma^2} \times \sqrt{Dn^2 - B\gamma^2} \times \overline{D^2m^2n^2 - BCa^2\gamma^2}} = \frac{\overline{Dm^2 - Ca^2} \times D^2m^2n^2\dot{z}}{\overline{D^2m^2n^2 - BCa^2\gamma^2}|^{\frac{3}{2}}}.$$

The meaſure of the angle of contact of the track of the momentary pole, in the *immoveable* ſpherical ſurface, with a great circle, will accordingly be

$e\dot{T} - \frac{\overline{Dm^2 - Ca^2} \times D^2m^2n^2\dot{z}}{\overline{D^2m^2n^2 - BCa^2\gamma^2}|^{\frac{3}{2}}} = \frac{\overline{b^2m^2 + c^2a^2 + d^2n^2} \,.\, \dot{z}}{\sqrt{D^2m^2n^2 - BCa^2\gamma^2}}$: by means of which meaſure we may deſcribe, by points, the track of the momentary pole in the ſpherical ſurface laſt mentioned.

There are other methods of finding that track; but I know none that is leſs difficult than this method, or in any reſpect more ſatisfactory.

The radius of the leſſer circle, which is the circle of curvature of the polar track in our immoveable ſpherical ſurface, will be =

$$\frac{a\dot{z}}{\sqrt{\dot{z}^2 + \text{ſq. of the meaſ. of the ang. of cont.}}} = \frac{\sqrt{D^2m^2n^2 - BCa^2\gamma^2}}{\sqrt{\overline{b^2 + c^2}|^2 \cdot m^2 + \overline{c^2 + d^2}|^2 \cdot n^2 - BC\gamma^2}}.$$

When B is = 0, or very ſmall in compariſon with D, and Dm^2 is leſs than Ca^2, the laſt mentioned radius will be equal, or nearly equal, to the invariable quantity $\frac{Dmn}{\sqrt{\overline{b^2 + d^2}|^2 \cdot m^2 + 4d^4n^2}}$; the track of the pole in the immoveable ſpherical ſurface being then exactly, or very nearly, a leſſer circle. At the ſame time, the polar track upon the moveable ſpherical ſurface will be exactly, or very nearly, a leſſer circle whoſe radius is m.

When C is $=0$, or very ſmall in compariſon with D, and Dm^2 is greater than Ca^2, the track of the pole in the immoveable ſpherical ſurface will be exactly, or very nearly, a leſſer circle whoſe radius is $= \frac{Dmn}{\sqrt{4c^4m^2+\overline{c^2+a^2}|^2 \, . \, n^2}}$; and then the polar track upon the moveable ſpherical ſurface will be exactly, or very nearly, a leſſer circle whoſe radius is n.

Whatever the curves may be which the momentary pole ſhall deſcribe in thoſe two ſpherical ſurfaces, the track upon the moveable ſurface will always touch and roll along the track in the immoveable ſurface (whilſt the common center of both ſurfaces remains at reſt), in the manner deſcribed in my Paper in the *Philoſophical Tranſactions* for the year 1777; the velocity of the point of contact being equal to the value of v computed above, which velocity when B is $=0$, or very ſmall in compariſon with D, and Dm^2 is leſs than Ca^2, will be exactly, or very nearly, $= \frac{d^2-b^2}{d^2+b^2} \times \frac{mne}{a^2}$; and when C is $=0$, or very ſmall in compariſon with D, and Dm^2 is greater than Ca^2, that velocity will be exactly, or very nearly, $= \frac{d^2-c^2}{d^2+c^2} \times \frac{mne}{a^2}$.

The polar track upon the moveable ſpherical ſurface will always roll along the convexity of the track in the immoveable ſpherical ſurface; the convexity or concavity of the former being turned towards the convexity of the latter, according as Dm^2 is greater or leſs than Ca^2. Which track in the immoveable ſpherical ſurface, when it is not circular, will touch a certain circle as often as γ, during the motion, ſhall become $=0$; and likewiſe another parallel circle as often as γ ſhall become equal to $\left.\overline{\frac{D}{C}}\right|^{\frac{1}{2}} \times m$, or $\left.\overline{\frac{D}{B}}\right|^{\frac{1}{2}} \times n$; the parts of the track between the points of contact being perfectly ſimilar. If Dm^2 be $= Ca^2$ (Dn^2 being then $= Ba^2$, and conſequently $\left.\overline{\frac{D}{C}}\right|^{\frac{1}{2}} \times m = \left.\overline{\frac{D}{B}}\right|^{\frac{1}{2}} \times n = a$), the ſaid track will make an infinite number of revolutions about a certain point, continually approaching nearer and nearer thereto, without arriving thereat in any finite time, though the length of the ſpiral ſo deſcribed cannot exceed a certain finite qnantity.

M. Euler has computed, that if the motive forces to turn the revolving body about AB, AC, AD, be reſpectively denoted by H, I, K;

$$\text{H will be} = \frac{M}{3} \cdot \frac{d^2+c^2}{a^3T} \times \text{flux. of } e\beta - \frac{M}{3a^5} \cdot Be^2\gamma\delta,$$

$$\text{I} \quad = \frac{M}{3} \cdot \frac{d^2+b^2}{a^3T} \times \text{flux. of } e\gamma + \frac{M}{3a^5} \cdot De^2\beta\delta,$$

$$\text{K} \quad = \frac{M}{3} \cdot \frac{c^2+b^2}{a^3T} \times \text{flux. of } e\delta - \frac{M}{3a^5} \cdot Ce^2\beta\gamma;$$

γ being ſuppoſed to decreaſe as T increaſes: and he has put the value of each of thoſe forces (H, I, K) $= 0$. In doing ſo, it ſeems to me, that he has erroneouſly aſſumed equations as generally true, which are only ſo in a particular caſe. For $\frac{M}{3a^5} \cdot Be^2\gamma\delta$ is the motive force to turn the body about AB, ariſing from the centrifugal force of its particles revolving about the momentary axis AP, ſuppoſing the pole to keep its place; and $\frac{M}{3} \cdot \frac{d^2+c^2}{a^3T} \times$ flux. of $e\beta$ is the value of the motive force requiſite to cauſe the *whole* variation of the velocity $\left(\frac{e\beta}{a}\right)$ about AB. But the firſt mentioned force *alone* does not, in general, cauſe *all* the variation of the velocity about AB; that velocity varies in conſequence of the evagation of the pole P; and that evagation is cauſed by the motive forces urging the body to turn about AB, AC, AD, *conjunctly*. Therefore the motive force $\frac{M}{3a^5} \cdot Be^2\gamma\delta$ about AB *only* will not, in general, be equal to $\frac{M}{3} \cdot \frac{d^2+c^2}{a^3\dot{T}} \times$ flux. of $e\beta$, the value of the *whole* motive force requiſite to cauſe the variation of the velocity $\frac{e\beta}{a}$, as M. Euler reckoned.

The like objection may, I conceive, be juſtly made to his other two equations ſimilar to that which is here particularly adverted to.

M. D'Alembert's radical errors, in treating this ſubject, appear to me nearly ſimilar to M. Euler's.

Other arguments may be adduced to prove, that the equations aſſumed by thoſe gentlemen are not well founded. If the forces to turn the body about the lines AB, AC, AD were each $= 0$, the velocities about thoſe lines muſt each remain invariable; but it ſeems abſolutely impoſſible that they can ever remain ſo, whilſt the angles which thoſe lines make with the momentary axis are each continually varying. Moreover, according to their concluſions, the tangent at P to the track of polar evagation, upon the moveable ſpherical ſurface, will not always be perpendicular to the direction in which the pole P will be urged to turn by the joint centrifugal force of the particles of the revolving body; whereas it is proved, I preſume, beyond a doubt, in my Paper above-mentioned, that the ſaid track will always be interſected at right angles by the direction in which the momentary pole ſhall, at any inſtant of time, be urged to turn by the force cauſing its evagation.

If we reſolve each of the three forces H, I, K, into two others; the one to turn the body about the diameter NAN, and the other to turn it about the momentary axis PAP, at right angles to that diameter; the forces to turn it in the laſt mentioned direction, ariſing from the ſaid forces H, I, K, will be

$$\frac{\beta H}{a} = \frac{M}{3} \cdot \frac{d^2+c^2}{a^4\dot{T}} \times \beta \text{ flux. of } e\beta - \frac{M}{3a^6} \cdot Be^2\beta\gamma\delta,$$

$$\frac{\gamma I}{a} = \frac{M}{3} \cdot \frac{d^2+b^2}{a^4\dot{T}} \times \gamma \text{ flux. of } e\gamma + \frac{M}{3a^6} \cdot De^2\beta\gamma\delta,$$

$$\frac{\delta K}{a} = \frac{M}{3} \cdot \frac{c^2+b^2}{a^4\dot{T}} \times \delta \text{ flux. of } e\delta - \frac{M}{3a^6} \cdot Ce^2\beta\gamma\delta.$$

The ſum of theſe forces, it is obvious, muſt be $=0$; the direction wherein they are ſuppoſed to act being at right angles to that in which the body will be actually urged to turn by the joint centrifugal force of its particles, and that being the only force whereby the motion of the body is ſuppoſed to be affected: which ſum $(B+C-D$ being $=0)$ is, when divided by $\frac{M}{3a^4\dot{T}}$,

$$= \left\{ \begin{array}{l} \overline{d^2+c^2} \cdot \beta^2\dot{e} + \overline{d^2+b^2} \cdot \gamma^2\dot{e} + \overline{c^2+b^2} \cdot \delta^2\dot{e} \\ \overline{d^2+c^2} \cdot e\beta\dot{\beta} + \overline{d^2+b^2} \cdot e\gamma\dot{\gamma} + \overline{c^2+b^2} \cdot e\delta\dot{\delta} \end{array} \right\} = 0.$$

But $\beta\dot{\beta}+\gamma\dot{\gamma}+\delta\dot{\delta}$ being before found $=0$, we have $\overline{d^2+c^2+b^2} \times \overline{\beta\dot{\beta}+\gamma\dot{\gamma}+\delta\dot{\delta}} = 0$; and $b^2\beta\dot{\beta}+c^2\gamma\dot{\gamma}+d^2\delta\dot{\delta}$ being alſo found $=0$; it evidently follows, that

$$\overline{d^2+c^2} \cdot \beta\dot{\beta} + \overline{d^2+b^2} \cdot \gamma\dot{\gamma} + \overline{c^2+b^2} \cdot \delta\dot{\delta} \text{ will be} = 0.$$

Therefore $\overline{d^2+c^2} \cdot \beta^2\dot{e} + \overline{d^2+b^2} \cdot \gamma^2\dot{e} + \overline{c^2+b^2} \cdot \delta^2\dot{e}$ will be $=0$: conſequently $\dot{e}$ will be $=0$, and e invariable; which agrees with what is ſaid above reſpecting the momentum of rotation.

The other forces ariſing by reſolution from the forces H, I, K, to turn the body about the diameter NAN, will be

$$-\frac{B\gamma\delta H}{S} = -\frac{M}{3a^3}\cdot\frac{d^2+c^2}{S\dot{T}}\cdot Be\gamma\delta\dot{\beta} + \frac{Me^2}{3a^5S}\cdot B^2\gamma^2\delta^2,$$

$$\frac{D\beta\delta I}{S} = \frac{M}{3a^3}\cdot\frac{d^2+b^2}{S\dot{T}}\cdot De\beta\delta\dot{\gamma} + \frac{Me^2}{3a^5S}\cdot D^2\beta^2\delta^2,$$

$$-\frac{C\beta\gamma K}{S} = -\frac{M}{3a^3}\cdot\frac{c^2+b^2}{S\dot{T}}\cdot Ce\beta\gamma\dot{\delta} + \frac{M}{3a^5S}\cdot C^2\beta^2\gamma^2;$$

$$S \text{ being} = \sqrt{D^2m^2n^2 - BCa^2\gamma^2}.$$

And, no external force being ſuppoſed to act on the body, it follows, that the ſum of theſe three forces muſt be = o: therefore we may infer, that

$\dot{T}$ will be $= \frac{a^2}{e} \times \frac{\overline{d^4-c^4}\,.\,\gamma\delta\dot{\beta} - \overline{d^4-b^4}\,.\,\beta\delta\dot{\gamma} + \overline{c^4-b^4}\,.\,\beta\gamma\dot{\delta}}{B^2\gamma^2\delta^2 + D^2\beta^2\delta^2 + C^2\beta^2\gamma^2}$; which agreeing with the value of $\dot{T}$ found above, the truth of our preceding proceſs is thus confirmed.

The force $\frac{Me^2}{3a^5S} \times \overline{B^2\gamma^2\delta^2 + D^2\beta^2\delta^2 + C^2\beta^2\gamma^2}$, ariſing from thoſe three forces, is the *whole* joint centrifugal force of the particles of the revolving body, to turn it about the diameter NAN the way it will actually be urged to turn by ſuch force; the value whereof ſo computed will be $\left(= \frac{Me^2}{3a^5} \times \sqrt{D^2m^2n^2 - BCa^2\gamma^2} = \frac{Me^2S}{3a^5}\right)$ equal to the value of the force F computed above, both being conſidered as urging the body to turn in the ſame direction. And the quantity

$$\frac{Me}{3a^3S\dot{T}} \times \overline{\overline{d^2+c^2}\,.\,\gamma\delta\dot{\beta} - \overline{d^2+b^2}\,.\,D\beta\delta\dot{\gamma} + \overline{c^2+b^2}\,.\,C\beta\gamma\dot{\delta}}$$

$\left(= \frac{e\dot{z}}{a^3\dot{T}}\right.$ × *the ſum of all the* $\left.w^2 \times p\right)$ is the value of the motive force which, acting in that very direction, is requiſite to cauſe the momentary pole to change its place as above deſcribed. Thus we ſee diſtinctly how the equation ariſes, by which the value of $\dot{T}$ is juſt now determined.

I do not find that the reſolving the forces H, I, K, in any other manner will conduce to the attainment of any uſeful concluſion.

It appears, by what is done above, that the force

$$\text{H is} = \frac{\text{M}e}{3a^3\text{R}\dot{\text{T}}} \times \text{CD}^2 m^2 \beta^2 \dot{\beta},$$

$$\text{I is} = \frac{\text{M}e}{3a^3\text{R}\dot{\text{T}}} \times \text{BC} \,.\, \overline{\text{C}n^2 - \text{B}m^2} \,.\, \gamma^2 \dot{\gamma},$$

$$\text{K} = \frac{-\text{M}e}{3a^3\text{R}\dot{\text{T}}} \times \text{BD}^2 n^2 \delta^2 \dot{\delta};$$

$$\text{R being} = \text{B}^2\gamma^2\delta^2 + \text{D}^2\beta^2\delta^2 + \text{C}^2\beta^2\gamma^2.$$

And it is obvious, that each of the three laſt mentioned forces will be $=0$, if any two of the quantities b, c, d, be equal; two of the values of thoſe forces then vaniſhing, by reaſon of that equality; and the third value alſo vaniſhing by either $\dot{\beta}$, $\dot{\gamma}$, or $\dot{\delta}$, being at the ſame time $=0$. Therefore, in that caſe it happens, that M. EULER's computation agrees with mine: in every other caſe, I am clearly of opinion, his concluſions are not true. The ſame may be ſaid of M. D'ALEMBERT's concluſions reſpecting the ſame propoſition.

The evagation of the pole of a revolving body conſidered above, does not ariſe from gravity, the attraction of any other body, or any external impulſe whatever; but is only the conſequence of the *inertia of matter*, and muſt neceſſarily enſue, according to the theory here explained, in every body in the univerſe, after having been made to revolve, without reſtraint, about any line paſſing through its center of gravity, that is not *a permanent* axis of rotation.

The Earth being neither uniformly denſe nor a perfect ſpheroid muſt, in ſtrictneſs, be conſidered as having only three permanent axes of rotation, agreeably to what I have proved in my *Mathematical Memoirs*; and, as it is diſturbed in its rotatory motion by the attraction of the ſun and moon (and other bodies in our ſyſtem); it follows; that it will not continually revolve about either of thoſe axes, but will revolve, or endeavour to revolve, about ſucceſſive momentary axes, as ſhewn above. If then its three permanent axes of rotation be called its *firſt*, *ſecond*, and *third* axes; and the poles of its *firſt* axis be thoſe about which its momentary poles are carried according to our theory; the *ſecond* and *third* axes will be in the plane of its equator, the three being at right angles to each other. Therefore, with reſpect to the above theory, this terreſtrial maſs muſt be conſidered of ſuch a form, that its equator, and any ſection parallel thereto, ſhall rather be elliptical than circular. And, denoting its firſt, ſecond, and third axes by b, c, d, reſpectively, obſervations evince, that the difference $c-b$ will be much greater than the difference $d-c$. Whence it follows, that (ſuppoſing the earth's rotatory motion to be diſturbed *only* by the centrifugal force ariſing from the *inertia* of its own particles) the track of polar evagation with us will be nearly circular, and the radius of the limiting circle very ſmall, whether we have regard to the moveable or immoveable ſpherical ſurface referred to above; but that, in the latter ſurface, ſuch circle will be much leſs than in the former: and it moreover follows, that the concavity of the track upon the moveable ſurface will continually touch and roll along the convexity of the track in the immoveable ſurface.

In other planets, the tracks of polar evagation may, from a ſimilar cauſe, be very different. The theory above explained evidently proves, that their axes of rotation may poſſibly vary greatly in poſition, merely through the *inertia of matter*; whilſt Providence has ſo ordered it, that the poſition of the axes of rotation of this planet ſhall, by that cauſe, be but very little altered.

* * *

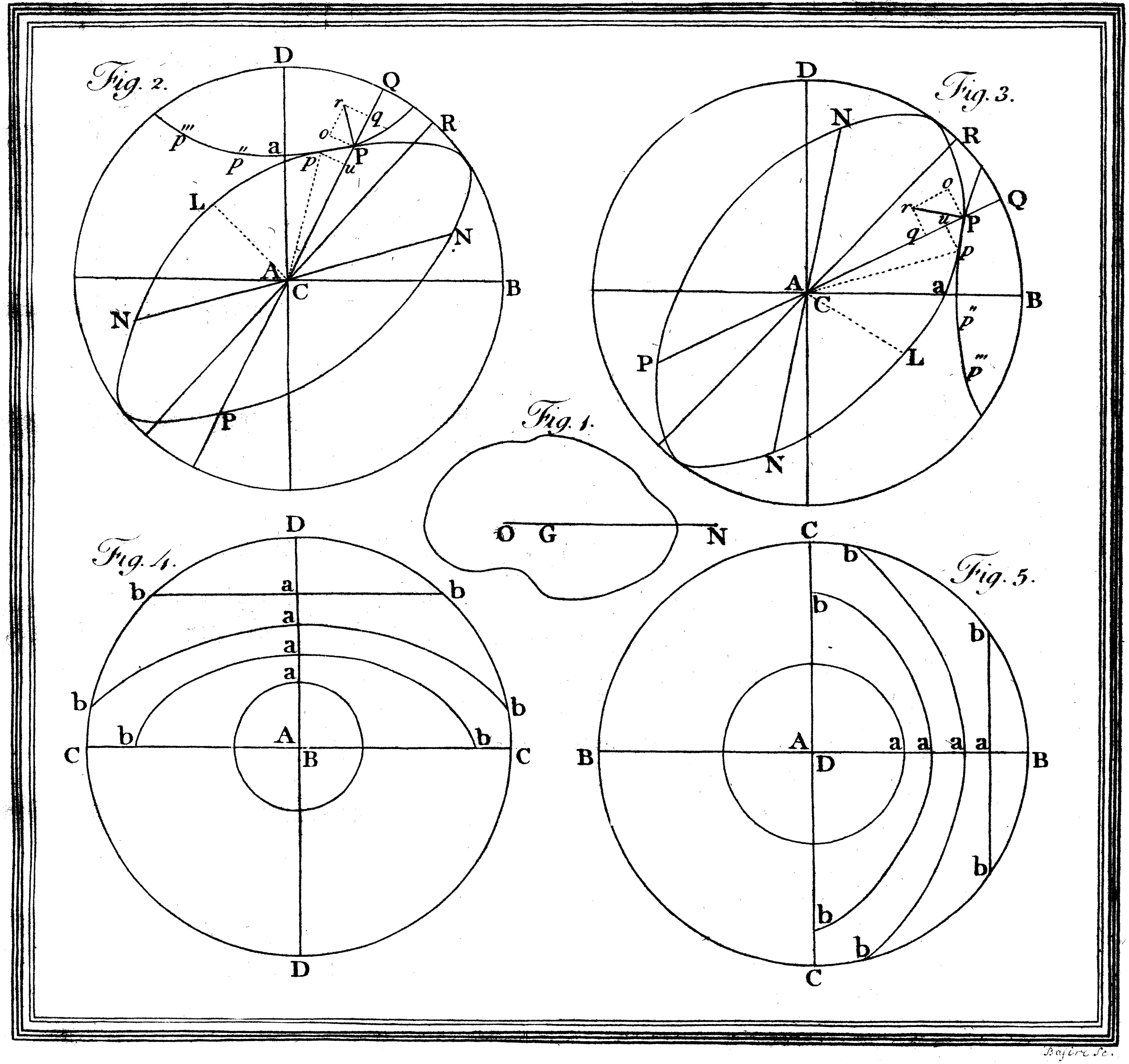

Fig. 2.

Fig. 3.

Fig. 1.

Fig. 4.

Fig. 5.

Basire sc.